Randomized Algorithms and Probabilistic Methods: Advanced Topics

Exercise 1
In this exercise, we prove that expander graphs are in some sense highly connected.
Prove that for every $\lambda < d$, there exists a constant $C$ such that the following holds for every $d$-regular graph $G$ on $n$ vertices with edge expansion at most $\lambda$: if one removes $m \geq 0$ edges from $G$, then the resulting graph has a connected component of size at least $n - Cm$.

Exercise 2
In this exercise, we prove that expander graphs have small diameter.
Prove that for every $\lambda < d$, there exists a constant $c$ such that for every $d$-regular graph $G$ with spectral expansion $\lambda$ and every vertex $v$ of $G$, one has
$$|B(v, r)| \geq \min\{(1 + c)^r, n\},$$
where $B(v, r)$ denotes the ball of radius $r$ around $v$ (i.e., the set of all vertices with distance at most $r$ to $v$). Hint: prove first the weaker statement that $|B(v, r)| \geq \min\{(1 + c)^r, n/2\}$. 