Randomized Algorithms and Probabilistic Methods: Advanced Topics

Solution to Exercise 1

(a) Assume $f$ is increasing and integrable on $[a, b]$. We can write
\[ \sum_{k=a}^{b} f(k) \leq \sum_{k=a}^{b-1} \int_{k}^{k+1} f(x) \, dx + f(b) = \int_{a}^{b} f(x) \, dx + f(b), \]
since $f(k) \cdot 1 \leq \int_{k}^{k+1} f(x) \, dx$ for increasing $f$. Furthermore,
\[ \sum_{k=a}^{b} f(k) \geq \sum_{k=a+1}^{b} \int_{k-1}^{k} f(x) \, dx + f(a) = \int_{a}^{b} f(x) \, dx + f(a), \]
since $f(k) \cdot 1 \geq \int_{k-1}^{k} f(x) \, dx$ for increasing $f$. The proof for decreasing $f$ is analogous and the statement follows.

(b) Let $H_n := \sum_{k=1}^{n} 1/k$. We can simply apply the result from (a) to get
\[ \ln n + \frac{1}{n} = \int_{1}^{n} \frac{1}{x} \, dx + \frac{1}{n} \leq H_n \leq \int_{1}^{n} \frac{1}{x} \, dx + 1 = \ln n + 1. \]
Looking closely at the error term in (a), we can actually obtain the stronger statement that the limit $\gamma = \lim_{n \to \infty} H_n - \ln n$ exists. The constant $\gamma$ is usually called the Euler-Mascheroni constant.

(c) Write
\[ \ln n! = \sum_{k=1}^{n} \ln k = \ln n + \int_{1}^{n} \ln x \, dx - \sum_{k=1}^{n-1} \int_{k}^{k+1} \ln x \, dx - \ln k. \]
Recall that $\int_{a}^{b} \ln x \, dx = b \ln b - a \ln a - a$. If we write $\Delta_k := \int_{k}^{k+1} \ln x \, dx - \ln k$, then this simplifies to
\[ \ln n! = \ln n + n \ln n - n + 1 - \sum_{k=1}^{n-1} \Delta_k. \quad (1) \]
We get
\[ \sum_{k=1}^{n-1} \Delta_k = \sum_{k=1}^{n-1} (k + 1) \ln (k + 1) - k \ln k - \ln k - 1 \]
\[ = \sum_{k=1}^{n-1} (k + 1)(\ln (k + 1) - \ln k) - 1 \]
\[ = \sum_{k=1}^{n-1} (k + 1)(1 + 1/k) - 1 \]
\[ = \sum_{k=1}^{n-1} (k + 1) \left(1 + \frac{1}{k} - \frac{1}{2k^2} + \mathcal{O}(k^{-3})\right) - 1 \]
\[ = \sum_{k=1}^{n-1} \frac{1}{2k} + \mathcal{O}(k^{-2}) \]
\[ = \frac{1}{2} \ln n + \mathcal{O}(1), \quad (2) \]
where in (2) we applied Taylor’s theorem, in (3) we applied (b), and in the last line, we used the well-known fact that $\sum_{k=1}^{\infty} 1/k^2$ converges. Plugging this into (1), we get

$$\ln n! = n \ln n - n + 1 + \frac{\ln n}{2} + O(1),$$

so

$$n! = e^{O(1)} \cdot \sqrt{n} \left(\frac{n}{e}\right)^n,$$

as required. Again, by doing the same proof a little more carefully, we obtain that there exists a constant $c$ such that $n! \sim c \cdot \sqrt{n} (n/e)^n$. Surprisingly, it turns out that $c = \sqrt{2\pi}$. This result is usually referred to as Stirling’s approximation to the factorial.

**Solution to Exercise 2**

For $a \leq 2$ we see that $\Delta(a) \leq 0.5$ because the number of missing trophies cannot be negative. Hence, for $a = 2$ we get $Y_{t+1} = 0$ if the team wins 2 or 3 trophies and for $a = 1$ we get $Y_{t+1} = 0$ if the team wins 1, 2 or 3 trophies. Similarly, for $a \geq 3$ we have $\Delta'(a) \geq 0.5$: for $a = 3$, if the team wins at 1, 2 or 3 trophies, then $Y_{t+1}$ is 0 and for $a = 4$, if the team wins 2 or 3 trophies, then $Y_{t+1}$ is 0. In both cases, $Y_t - Y_{t+1} \geq 3$.

**Solution to Exercise 3**

Equation (1.2) from the lecture notes is equivalent to

$$E[X_{t+1} | X_t] \leq (1 - \delta)E[X_t | X_t] = (1 - \delta)X_t,$$

by linearity of expectation. Taking the expectation on both sides,

$$E[E[X_{t+1} | X_t]] \leq (1 - \delta)E[X_t].$$

Since $E[E[X_{t+1} | X_t]] = E[X_{t+1}]$, we get $E[X_{t+1}] \leq (1 - \delta)E[X_t]$, and so, by induction, we obtain

$$E[X_t] \leq (1 - \delta)^t E[X_0] = (1 - \delta)^t s_0,$$

as required.

**Solution to Exercise 4**

Let $(X_t)_{t \geq 0}$ be a (time-homogeneous) Markov chain with state space $0 \in S \subseteq \mathbb{R}_0^+$ such that the values $Y(x) = E[T | X_0 = x]$ are well-defined. Note that $Y(a) = 0$ if and only if $a = 0$.

For all $x_0 \neq 0$, we have

$$Y(x_0) = E[T | X_0 = x_0] = \sum_{x_1 \in S} \Pr[X_1 = x_1 | X_0 = x_0] \cdot E[T | X_1 = x_1]$$

$$= \sum_{x_1 \in S} \Pr[X_1 = x_1 | X_0 = x_0] \cdot (1 + E[T | X_0 = x_1])$$

$$= 1 + E[Y_1 | X_0 = x_0].$$

Thus, for all $a \neq 0$, we have

$$E[Y_{t+1} | Y_t = a] = E[Y_1 | Y_0 = a] = \sum_{x \in S} \Pr[X_0 = x | Y_0 = a] \cdot E[Y_1 | X_0 = x] = a - 1,$$

so

$$E[Y_1 - Y_{t+1} | Y_t = a] = 1,$$

as required.