Solution to Exercise 1

We divide the process into two phases: the first phase when there are still isolated vertices, and the second phase when there are no such vertices any more.

**First phase.** Our first goal is to estimate the duration of the first phase. Let $Z_0(m)$ denote the number of isolated vertices in $G_m$. We have $Z_0(0) = n$ and for $x > 0$ we have

$$E[Z_0(m + 1) - Z_0(m) \mid Z_0(m) = x] = -1 - x/n,$$

since we will connect an isolated vertex with a randomly chosen other vertex, which is isolated with probability $x/n$. Thus, for $z_0(t)$ being the solution of $z_0(0) = 1$ and

$$z_0(t) = -1 - z_0(t),$$

the theorem from the lecture tells us that whp

$$Z_0(tn) = nz_0(t) + o(n)$$

holds for all $t$ such that $Z_0(t) > 0$.

We can solve the differential equation (1) using the method of variation of constants. The solution to the homogeneous system $x'(t) = -x(t)$ is $x(t) = e^{-t}$. Assuming now $z_0(t) = c(t)e^{-t}$, we obtain $z_0'(t) = c'(t)e^{-t} - z_0(t)$. To match (1) we need $c'(t)e^{-t} = -1$ and $c(0)e^{-0} = 1$, i.e., $c(t) = 2 - e^t$. Therefore

$$z_0(t) = (2 - e^t)e^{-t} = 2e^{-t} - 1$$

is the unique solution of (1).

In particular, as $2e^{-t} - 1 = 0$ only in $t = \ln 2$, we obtain that the time when all isolated vertices are gone is close to $n \ln 2$.

To study the next phase, we also need to know the number of degree-one vertices at the end of the first phase. This can also be obtained with differential equations. First, let $Z_1(m)$ denote the number of vertices of degree one in $G_m$. We have $Z_1(0) = 0$ and for $x_0 > 0,$

$$E[Z_1(m + 1) - Z_1(m) \mid Z_1(m) = x_1 \text{ and } Z_0(m) = x_0] = 1 + x_0/n - x_1/n,$$

since every round a vertex goes from being isolated to having degree one, and there is a probability of $x_0/n$ to turn an additional degree zero vertex to degree one, and a probability of $x_1/n$ to turn a vertex with degree one into a vertex with degree two. We thus get the differential equation $z_1(0) = 0$ and $z_1'(t) = 1 + z_0(t) - z_1(t) = 2e^{-t} - z_1(t)$. Again the solution is obtained by variation of constants. The homogeneous system has the solution $e^{-t}$, so we use the Ansatz $z_1(t) = c(t)e^{-t}$, giving $z_1'(t) = c'(t)e^{-t} - z_0(t)$. This time we want $c'(t)e^{-t} = 2e^{-t}$ and $c(0)e^{-0} = 0$, i.e., $c(t) = 2t$. Thus we have

$$Z_1(tn) = nz_1(t) + o(n) = n2te^{-t},$$

which is valid for all $t$ such that $Z_0(tn) > 0$.

In particular, since the first phase whp ends around time $n \ln 2$, we get that at the beginning of the second phase, there are $n \ln 2 + o(n)$ many vertices of degree one whp.
**Second phase.** In the second phase, we start with no isolated vertices and \( n \ln 2 + o(n) \) vertices of degree one. For \( x > 0 \) we have

\[
E[Z_1(m + 1) - Z_1(m) \mid Z_1(m) = x \text{ and } Z_0(m) = 0] = -1 - x/n.
\]

From this, we get the differential equation \( z_1(t) = \ln 2 \) and \( z'_1(t) = -1 - z_1(t) \). The solution can be determined as before to be

\[
z_1(t) = (1 + \ln 2)e^{-t} - 1.
\]

The second phase ends whp when \( z_1(t) = 0 \), i.e., when

\[
t = \ln(1 + \ln 2).
\]