Solution to Exercise 1

Let $R_t$ denote the number of red balls at time $t$, let $B_t$ denote the number of blue balls at time $t$, and let $X_t := R_t - n$. Initially, $X_0 = s_0 := n$. Let $T$ be the smallest integer $t > 0$ for which $X_t = 0$. We need to compute the drift

$$ \Delta(a) := \mathbb{E}[X_t - X_{t+1} | X_t = a]. $$

Consider a fixed round $t < T$. Then

$$ \mathbb{E}[R_t - R_{t+1} | R_t] = \frac{R_t}{2n} - \frac{B_t}{2n} = \frac{X_t + n - (2n - (X_t + n))}{2n} = \frac{X_t}{n}, $$

so $\Delta(a) = a/n$.

By the multiplicative drift theorem, $\mathbb{E}[T | X_0 = n] \leq n \ln n + n$ and

$$ \Pr[T > [n \ln n + cn]] \leq e^{-c}. $$

Solution to Exercise 2

Let $X_t$ be the number of pictures that Alice does not own an odd number of times. Initially, $X_0 = n$. Let $T$ be the first round for which $X_t = 0$. The drift of $X_t$ is

$$ \Delta(a) = \mathbb{E}[X_t - X_{t+1} | X_t = a] = \frac{a(a - 1)}{n^2} + \frac{a}{2n} $$

for all $a > 0$.

Let $c(x) = x - 2$. Then we have $X_t \geq c(X_{t+1})$. Moreover, for $a > 0$,

$$ \Delta(a) \leq h(c(a)), $$

where $h(x) = \Delta(x + 2)$.

We have

$$ \frac{1}{\Delta(a)} = \frac{2n^2}{2a(a - 1) + na} = \frac{2n^2}{a(2a + n - 2)} = \frac{2n^2}{a(n - 2)} - \frac{4n^2}{(2a + n - 2)(n - 2)} $$

By the lower-bound version of the variable drift theorem,

$$ \mathbb{E}[T] \geq \int_1^n \frac{dx}{h(x)} = \int_3^{n+2} \frac{dx}{\Delta(x)} = \frac{2n^2}{n - 2} \left[ \ln(x) - \ln(x + n/2 - 1) \right]_3^{n+2}. $$

Therefore,

$$ \mathbb{E}[T] \geq \frac{2n^2}{n - 2} \left( \ln(n + 2) - \ln 3 - \ln(n + 2 + n/2 - 1) + \ln(n/2 + 2) \right) = 2n \ln n + O(n). $$

The upper bound is similar.
Solution to Exercise 3

Let \( g(x) = \ln x \ln \ln x \) and define

\[
Y_t = \begin{cases} 
   g(X_t) & \text{if } X_t \geq e, \\
   0 & \text{otherwise.}
\end{cases}
\]

Let \( T \) be the first point in time where \( Y_T = 0 \). Note that this is also the first point in time where \( X_T \) goes below \( e \). Thus it suffices to show that \( \mathbb{E}[T] = \Omega(\ln n \ln \ln n) \).

(a) Let us compute the drift

\[
\Delta(a) = \mathbb{E}[Y_t - Y_{t+1} | Y_t = a].
\]

Fix any \( a > 0 \) and let \( x = g^{-1}(a) \). We have

\[
\Delta(a) = a - \sum_{i=3}^{\left\lfloor ex \right\rfloor} \frac{g(i)}{1 + \left\lfloor ex \right\rfloor},
\]

by conditioning on the different possible values of \( X_{t+1} \). We can bound

\[
\sum_{i=3}^{\left\lfloor ex \right\rfloor} g(i) \geq \int_e^{ex} g(z) \, dz - g(ex)
\]

and

\[
\frac{1}{1 + \left\lfloor ex \right\rfloor} \geq \frac{1}{2 + ex} \geq \frac{1}{ex} - \frac{2}{e^2 x^2}.
\]

Combining everything,

\[
\Delta(a) \leq a - \left( \frac{1}{ex} - \frac{2}{e^2 x^2} \right) \left( \int_e^{ex} g(z) \, dz - g(ex) \right) = a - \frac{1}{ex} \int_e^{ex} g(z) \, dz + o(1),
\]

as \( x \to \infty \).

The next step is to bound the integral of \( g(z) \). Let \( a(z) = z \ln z - z \) and let \( b(x) = \ln \ln z \). Observe that \( a'(z) = \ln z \). Integrating by parts,

\[
\int_e^{ex} g(z) \, dz = \int_e^{ex} a'(z)b(z) \, dz = [a(z)b(z)]_e^{ex} - \int_e^{ex} a(z)b'(z) \, dz.
\]

Since \( a(z)b'(z) = (z \ln z - z)/(z \ln z) = 1 - 1/\ln z \), we can simply upper bound

\[
\int_e^{ex} a(z)b'(z) \, dz \leq \int_e^{ex} 1 \, dx \leq ex.
\]

Therefore, we obtain

\[
\int_e^{ex} g(z) \, dz \geq (ex \ln(ex) - ex) \ln \ln(ex) - ex = ex \cdot g(ex) - ex \ln \ln x - ex.
\]

Plugging this into (1), we get

\[
\Delta(a) \leq a - g(ex) + \ln \ln x + 1 + o(1).
\]

Since

\[
g(ex) = (1 + \ln x) \ln(1 + \ln x) = \ln \ln x + \ln x \ln \ln x + o(1),
\]

and \( a = g(x) = \ln x \ln \ln x \), this gives

\[
\Delta(a) \leq 1 + o(1).
\]

(b) Let \( C > 0 \) be such that \( \Delta(a) < C \) for all \( a > 0 \). The Theorem 1.1 immediately gives

\[
\mathbb{E}[T] \geq Y_0/C = g(n)/C = (\ln n \ln \ln n)/C,
\]

under the assumption that \( \lim_{t \to \infty} \mathbb{E}[X_t] = 0 \). 


(c) Let $C > 0$ be such that $\Delta(a) < C$ for all $a > 0$. Then, as in the proof of Theorem 1.1, we have

$$\Pr[T > t] \geq \frac{\mathbb{E}[Y_t - Y_{t+1}]}{C}.$$ 

Therefore

$$C \mathbb{E}[T] \geq \sum_{t=0}^{\infty} C \Pr[T > t] \geq \sum_{t=0}^{t_0} C \Pr[T > t] \geq \sum_{t=0}^{t_0} \mathbb{E}[Y_t] - \mathbb{E}[Y_{t+1}] = \mathbb{E}[Y_0] - \mathbb{E}[Y_{t_0+1}].$$

In particular (taking $\lim \sup$ on both sides),

$$C \mathbb{E}[T] \geq \mathbb{E}[Y_0] - \liminf_{t \to \infty} \mathbb{E}[Y_t].$$

Since $Y_0 = \ln n \ln \ln n$, the claim follows.

(d) Assume that $\liminf_{t \to \infty} \mathbb{E}[Y_t] > 0$. Then there must exist some $\delta$ and some positive integer $t_0$ such that for all $t \geq t_0$, we have $\mathbb{E}[Y_t] \geq \delta$.

We have

$$\Pr[Y_t > 0] = \frac{\mathbb{E}[Y_t]}{\mathbb{E}[Y_t | Y_t > 0]}.$$ 

Note that, deterministically,

$$Y_t \leq g(ne^t) = \ln(ne^t) \ln \ln(ne^t) = (t + \ln n) \cdot (\ln(t + \ln n)).$$

Therefore, if $t_0$ is sufficiently large, then for all $t \geq t_0$,

$$\Pr[Y_t > 0] \geq \frac{\delta}{(2t \ln(2t)).}$$

From this, we obtain

$$\mathbb{E}[T] = \sum_{t=0}^{\infty} \Pr[T > t] = \sum_{t=0}^{\infty} \Pr[Y_t > 0] \geq \sum_{t=t_0}^{\infty} \frac{\delta}{(2t \ln(2t))} = \infty.$$