Exercise 1:
Given a dynamic weighted graph $G(V, E)$, where $|V| = n$ and $|E| = m$, we attempt to extend the fully dynamic connectivity data structure to build a dynamic minimum spanning tree data structure that allows only deletion operations. Recall the key ingredients for the fully dynamic connectivity as follows

- Assign each edge a level that starts at $\log n$ but only decreases over time.
- Let $G_i$ be the subgraph of $G$ consisting of edges with level at most $i$, i.e., $G_0 \subseteq G_1 \subseteq \cdots \subseteq G_{\log n} = G$.
- Let $F_i$ be a spanning forest of $G_i$ for $1 \leq i \leq \log n$, and let $F$ be $F_{\log n}$.

Two invariants are required for fully dynamic connectivity:

**Invariant 1.** Every connected component $G_i$ has at most $2^i$ vertices.

**Invariant 2.** $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{\log n}$, and $F$ is a minimum spanning tree/forest of $G$ if the level of an edge is interpreted as its weight.

If the deletion of an edge $e = (u, v)$ separates a tree $T \in F$ into two subtree $T_v$ and $T_u$, then we need to find the lightest edge connecting $T_v$ and $T_u$ as the replacement edge. Therefore, another invariant is suggested:

**Invariant 3.** Every cycle $C$ has a non-tree edge of maximum weight and maximum level among all the edges in $C$.

Please complete the following two tasks:

- Prove that among all the replacement edges, the lightest edge is on the minimum level.
- Assume the level of $e$ to be $\ell$, and describe how to find the replacement edge.

**Hint:** Consider two replacement edge $e_1$ and $e_2$ where the weight of $e_1$ is larger than the weight $e_2$. Before the deletion of $e$, inserting $e_1$ (resp. $e_2$) into $F$ will form a cycle $C_1$ (resp. $C_2$). Compare the levels of $e_1$ and $e_2$ using Invariant 3 and the cycle $C = C_1 \cup C_2 \setminus C_1 \cap C_2$.

Exercise 2:
Please prove the lower bound of a deletion operation for the dynamic minimum spanning tree data structure to be $\Omega(\log n)$.

**Hint:**
- Reduce the standard sorting problem to a sequence of deletion operations.