Algorithmic Game Theory	Summer 2015, Week 1
Introduction to Congestion Games	
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Our first topic will be congestion games. We define the concepts of (pure) Nash equilibrium and best response. We show that every congestion game has a pure Nash equilibrium, and investigate the number of (best response) improvement steps needed to reach a pure Nash equilibrium.

1 Definitions and Preliminaries

Definition 1.1 (Congestion Game (Rosenthal 1973))**.** *A* congestion game *is a tuple* $\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}})$ *with*

- $\mathcal{N} = \{1, \ldots, n\}$*: set of players*
- $\mathcal{R} = \{1, \ldots, m\}$*: set of resources*
- $\Sigma_i \subseteq 2^{\mathcal{R}}$*: strategy space of player* i
- $d_r \colon \{1, \ldots, n\} \to \mathbb{Z}$*: delay function for resource* r

For any state $S = (S_1, \ldots, S_n) \in \Sigma_1 \times \cdots \Sigma_n$*,*

- $n_r(S) = |\{i \in \mathcal{N} \mid r \in S_i\}|$*: number of players with* $r \in S_i$
- $d_r(n_r(S))$*: delay of resource* r
- $\delta_i(S) = \sum_{r \in S_i} d_r(n_r(S))$*: delay of player* i

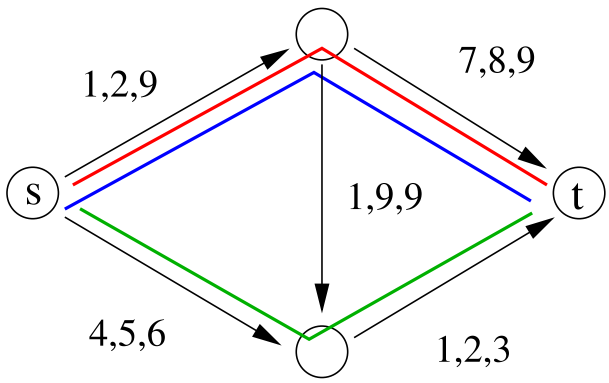
The cost *of player* i *in state* S *is* $c_i(S) = \delta_i(S)$*, that is, players aim at minimizing their delays.*

We denote by $S_{-i} = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ the states of all players other than $i \in \mathcal{N}$. This notation allows us to concisely define a unilateral deviation of a player. For $i \in \mathcal{N}$ let $S \in \Sigma$ and $S'_i \in \Sigma_i$, then $(S'_i, S_{-i}) = (S_1, \ldots, S_{i-1}, S'_i, S_{i+1}, \ldots, S_n)$.

Definition 1.2. *A strategy* S_i *is called a* best response *for player* $i \in \mathcal{N}$ *against a collection of strategies* S_{-i} *if* $c_i(S) \leq c(S'_i, S_{-i})$ *for all* $S'_i \in \Sigma_i$*.*

Definition 1.3. *A state* $S \in \Sigma$ *is called a* (pure) Nash equilibrium *if for every player* $i \in \mathcal{N}$*,* S_i *is a best response against* S_{-i}*.*

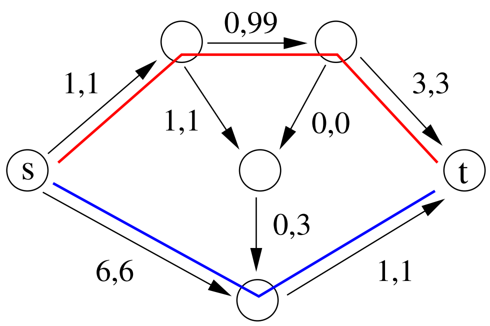
Example 1.4 (Network Congestion Game)**.** *Given a directed graph* $G = (V, E)$ *with delay functions* $d_e \colon \{1, \ldots, n\} \to \mathbb{Z}$*,* $e \in E$*. Player* i *wants to allocate a path of minimal delay between a source* $s_i \in V$ *and a target* $t_i \in V$*.*



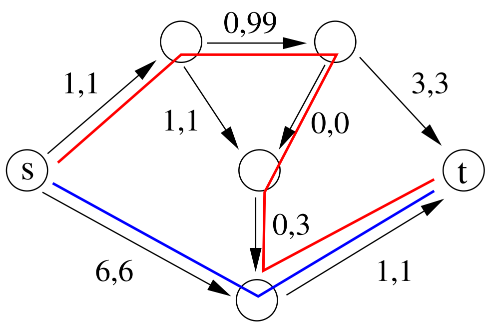
In this example, $\mathcal{N} = \{1,2,3\}$, $\mathcal{R} = E$, Σ_i = set of s-t paths.
This game is symmetric, *i.e., all players have the same set of strategies.*

Example 1.5. *A sequence of (best response) improvement steps:*

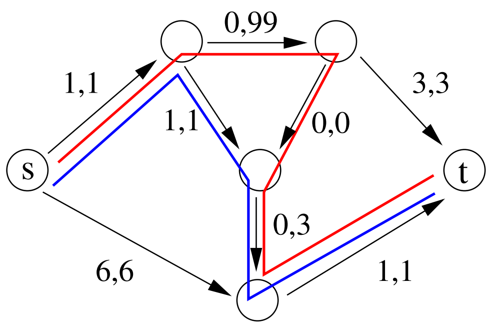
start:



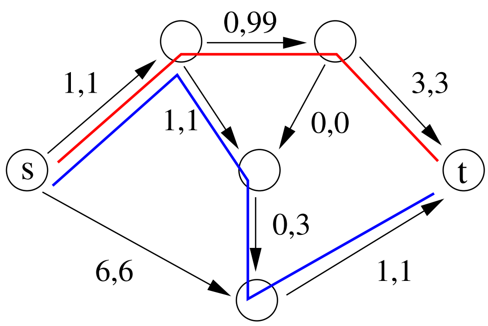
after first improvement (red player):



after second improvement (blue player):



after third improvement (red player):



reached pure nash equilibrium

Questions

- Does every congestion game posses a pure Nash equilibrium?
- Is every sequence of improvement steps finite?
- How many steps are needed to reach a (pure) Nash equilibrium?
- What is the complexity of computing (pure) Nash equilibria in congestion games?

2 Existence of Pure Nash Equilibria

Theorem 1.6 (Rosenthal 1973)**.** *For every congestion game, every sequence of improvement steps is finite.*

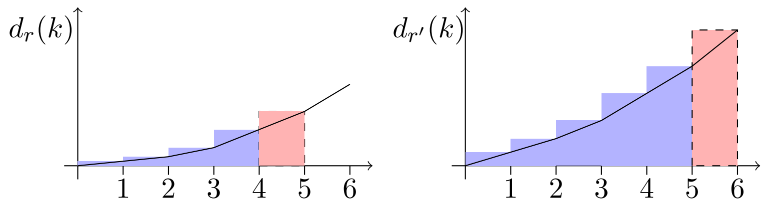


Figure 1: Proof of Lemma 1.8: The contribution of two resources r and r' to the potential is the shaded area. If a player changes from r' to r, his delay changes exactly as the potential value (difference of red areas).

This result immediately implies

Corollary 1.7. *Every congestion game has at least one pure Nash equilibrium.*

Proof of Theorem 1.6. Rosenthal's analysis is based on a potential function argument. For every state S, let

$$\Phi(S) = \sum_{r \in \mathcal{R}} \sum_{k=1}^{n_r(S)} d_r(k) \enspace .$$

This function is called *Rosenthal's potential function.*

Lemma 1.8. *Let S be any state. Suppose we go from S to a state S' by an improvement step of player i decreasing his delay by $\Delta > 0$. Then $\Phi(S') = \Phi(S) - \Delta$.*

Proof. The potential $\phi(S)$ can be calculated by inserting the players one after the other in any order, and summing the delays of the players at the point of time at their insertion.

Without loss of generality player i is the last player that we insert when calculating $\Phi(S)$. Then the potential accounted for player i corresponds to the delay of player i in state S. When going from S to S', the delay of i decreases by Δ, and, hence, Φ decreases by Δ as well (see Figure 2 for an example.) □

The lemma shows that Φ is a so-called *exact potential*, i.e., if a single player decreases its latency by a value of $\Delta > 0$, then Φ decreases by exactly the same amount.

Further observe that

(i) the delay values are integers so that, for every improvement step, $\Delta \geq 1$,

(ii) for every state S, $\Phi(S) \leq \sum_{r \in \mathcal{R}} \sum_{i=1}^{n} |d_r(i)|$,

(iii) for every state S, $\Phi(S) \geq -\sum_{r \in \mathcal{R}} \sum_{i=1}^{n} |d_r(i)|$.

Consequently, the number of improvements is upper-bounded by $2 \cdot \sum_{r \in \mathcal{R}} \sum_{i=1}^{n} |d_r(i)|$ and hence finite. □

3 Convergence Time of Improvement Steps

Rosenthal's theorem shows that any sequence of improvement steps is finite. However, it does not give any guarantee how many improvement steps are needed to reach a Nash equilibrium.

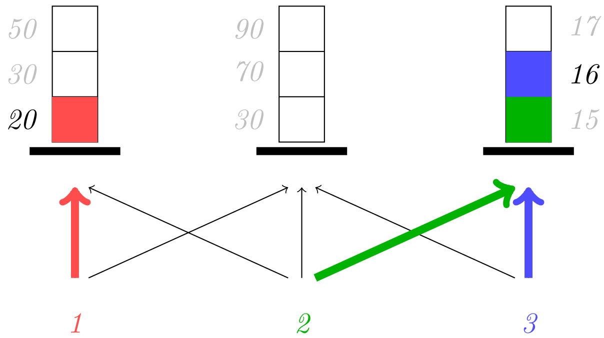
A trivial upper bound on the length of any (finite) sequence of improvement steps is the overall number of states, which is at most 2^{mn}. However, this is only a very poor guarantee and by no means tight.

We will show a significantly better, namely polynomial, guarantee for *singleton congestion games*. In this subclass of congestion games every player wants to allocate only a single resource at a time from a subset of allowed resources. Formally:

Definition 1.9 (Singleton Games)**.** *A congestion game is called* singleton *if, for every* $i \in \mathcal{N}$ *and every* $R \in \Sigma_i$*, it holds that* $|R| = 1$*.*

Although this constraint on the strategy sets is quite restrictive, there are still up to m^n different states.

Example 1.10 (Singleton Congestion Game)**.** *Consider a "server farm" with three servers a, b, c (resources) and three players 1,2,3 each of which wants to access a single server.*



The colored arrows indicate a pure Nash equilibrium.

Theorem 1.11. *In a singleton congestion game with* n *players and* m *resources, all improvement sequences have length* $O(n^2 m)$*.*

Proof idea:

- Replace original delays by bounded integer values without changing the preferences of the players.
- Show an upper bound on the maximum potential with respect to new delays.
- Due to integer values, decrease of potential in an improvement step is at least 1. Hence, length of every improvement sequence is bounded by maximum potential.

Proof. Sort the set of delay values $V = \{d_r(k) \mid r \in \mathcal{R}, 1 \leq k \leq n\}$ in increasing order. Define alternative, new delay functions:

$$\bar{d}_r(k) := \text{position of } d_r(k) \text{ in sorted list.}$$

The new delay of a player i using resource r in state S is $\bar{\delta}_i(S) = \bar{d}_r(n_r(S))$.

Observation 1.12. *Let* S *and* S' *be two states such that* (S, S') *is an improvement step for some player* i *with respect to the original delays. Then* (S, S') *is an improvement step for* i *with respect to the new delays, as well.*

Furthermore, observe that $\bar{d}_r(k) \leq nm$ for all $r \in \mathcal{R}$ and $k \in [n]$ because there are at most nm elements in V. Therefore, Rosenthal's potential function with respect to the new delays $\bar{d}_r(k)$ can be upper-bounded as follows:

$$\bar{\Phi}(S) = \sum_{r \in \mathcal{R}} \sum_{k=1}^{n_r(S)} \bar{d}_r(k) \leq \sum_{r \in \mathcal{R}} \sum_{k=1}^{n_r(S)} n\, m \leq n^2\, m \ .$$

It holds that $\bar{\Phi} \geq 1$. Also, $\bar{\Phi}$ decreases by at least 1 in every step. Therefore, the length of every improvement sequence is upper-bounded by $n^2 m$. $\square$

Example 1.13. *The sorted list of delay values in Example 1.10 is*

$$15, 16, 17, 20, 30, 50, 70, 90.$$

Hence, the old and new delay functions are

$$\begin{aligned} d_a(1,2,3) &= (20,30,50) \qquad & \bar{d}_a(1,2,3) &= (4,5,6) \\ d_b(1,2,3) &= (30,70,90) \qquad & \bar{d}_b(1,2,3) &= (5,7,8) \\ d_c(1,2,3) &= (15,16,17) \qquad & \bar{d}_c(1,2,3) &= (1,2,3) \end{aligned}$$

Recommended Literature

- D. Monderer, L. Shapley. Potential Games. Games and Economic Behavior, 14:1124–1143, 1996. (Equivalence congestion and potential games)
- H. Ackermann, H. Röglin, B. Vöcking. On the impact of combinatorial structure on congestion games. Journal of the ACM, 55(6), 2008. (Generalization of theorem on singleton games)