Algorithmic Game Theory — Summer 2015, Week 11

Non-Truthful Mechanisms Beyond the Worst-Case

Or: How Google Got So Incredibly Rich

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We have already mentioned that most search engines do *not* use the truthful Vickrey-Clarke-Groves Mechanism (VCG). They use a different, non-truthful mechanism called Generalized Second-Price (GSP). While the VCG mechanism has a dominant strategy equilibrium which maximizes social welfare, one can use the tools that we developed last week to show that the GSP mechanism has a small Price of Anarchy.

This comparison alone, however, does not justify the use of GSP over VCG. In fact, the stronger properties of VCG could rather be used to argue that the VCG mechanism should be used. In this lecture we will complement the Price of Anarchy framework with additional tools that enable a more fine-grained analysis. Just as in the case of potential games we will consider the Price of Stability and equilibrium refinements.

1 Sponsored Search Auctions

We begin by recalling our definition of a sponsored search auction and how the VCG mechanism for this setting looks like. We also define the GSP mechanism, which is used by most search engines in practice.

Definition 11.1 (Sponsored Search Auction)**.** *There are n bidders competing for the assignment of one of k slots. Each bidder i has a (private) value-per-click v_i and each slot j has a (known) click-through-rate α_j. We assume that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Bidder i's value for slot j is given by the product $\alpha_j \cdot v_i$.*

Definition 11.2 (VCG Mechanism)**.** *The Vickrey-Clarke-Groves Mechanism for sponsored search auctions proceeds as follows:*

1. *Collect a bid b_i from each agent $i \in \{1, \dots, n\}$.*
2. *Sort bidders such that $b_1 \geq b_2 \geq \cdots \geq b_n$.*
3. *For $i = 1, \dots, k$: Assign bidder i to slot i and make him pay $P_i(b) = \sum_{\ell=i}^{k} (\alpha_\ell - \alpha_{\ell+1}) \cdot b_{\ell+1}$.*

Definition 11.3 (GSP Mechanism)**.** *The Generalized Second-Price Mechanism for sponsored search auctions proceeds as follows:*

1. *Collect a bid b_i from each agent $i \in \{1, \dots, n\}$.*
2. *Sort bidders such that $b_1 \geq b_2 \geq \cdots \geq b_n$.*
3. *For $i = 1, \dots, k$: Assign bidder i to slot i and make him pay $P_i(b) = \alpha_i \cdot b_{i+1}$.*

Note that the payments under GSP are point-wise higher than under VCG. That is, for the same vector of bids b the payments $P_i(b)$ of each bidder i in the GSP mechanism are at least as high as in the VCG mechanism.

The utility of bidder i in a given mechanism is $u_i(b) = \alpha_j \cdot v_i - P_i(b)$. For the GSP mechanism we use $p_i(b) = b_{i+1}$ or simply p_i to denote the payment per click. Truthfulness and the basic non-truthful equilibrium concepts are defined as before. The *social welfare* is $SW(b) = \sum_{i=1}^{k} \alpha_i \cdot v_i$ and the *revenue* is $R(b) = \sum_{i=1}^{k} P_i(b)$.

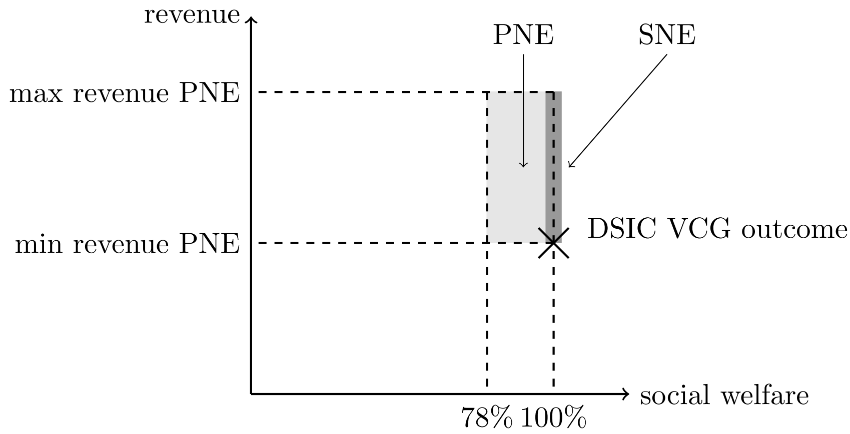

Figure 1: Overview of results. Pure Nash equilibria are light gray, while symmetric pure Nash Nash equilibria are dark gray.

It is not difficult to see that the GSP mechanism unlike the VCG mechanism is not dominant strategy incentive compatible.

Example 11.4. *There are three bidders and three slots. The click-through rates are* $\alpha = (1, 0.7, 0.1)$ *and values are* $20, 10, 5$. *Suppose the bidder with value* 20 *faces bids* 10 *and* 5. *A truthful bid gives him slot* 1 *for a utility of* $1.0 \cdot (20 - 10) = 10$. *Any bid between* 10 *and* 5 *would give him slot* 2 *for a utility of* $0.7 \cdot (20 - 5) = 10.5$.

However, it turns out that there is always a pure Nash equilibrium of the GSP mechanism, which maximizes social welfare and in which every bidder pays what he would pay in the truthful VCG equilibrium.

Example 11.5. *Suppose the bidder with value* 20 *bids* 10, *the bidder with value* 10 *bids* 6, *and the bidder with value* 5 *bids* $30/7 \approx 4.286$. *Then the highest value bidder wins the first slot at a price of* 6, *the second highest value bidder wins the second slot at a price of* 3, *and the third and lowest value bidder wins the third slot for free.*

On the other hand not all pure Nash equilibria of the GSP mechanism maximize social welfare as the following example shows.

Example 11.6. *Suppose the bidder with value* 20 *bids* 6, *the bidder with value* 10 *bids* 10, *and the bidder with value* 5 *bids* $30/7 \approx 4.26$. *Then the second highest value bidder wins the first slot, the highest value bidder wins the second slot, and the third and lowest value bidder wins the third slot. Note that the social welfare is* 24.5, *which is a factor* ≈ 1.127 *smaller than the optimal social welfare of* 27.5.

In fact, this cursory analysis is not too far off from the truth (see Figure 1). The Price of Anarchy with respect to pure Nash equilibria is known to be at most 1.282 and the Price of Stability is 1.[1] Instead of proving these results we will focus on a *refinement* of pure Nash equilibria—symmetric pure Nash equilibria—and show that for these equilibria $PoA = PoS = 1$. We will also show that the smallest revenue in any such equilibrium coincides with the revenue in the truthful VCG equlibrium and the largest revenue coincides with the maximum revenue in any pure Nash equilibrium.

[1] The Price of Anarchy bound for pure Nash equilibria is almost tight. There is an example with three bidders in which the Price of Anarchy is 1.259. For more general solution concepts such as coarse correlated equilibria the bound is slightly worse but still constant.

2 A Refinement of Pure Nash Equilibria

Our focus will be on so-called symmetric pure Nash equilibria. Symmetric pure Nash equilibria are obtained from pure Nash equilibria by removing the "asymmetry" in the equilibrium conditions. This removal in fact strengthens the equilibrium concept and so every symmetric pure Nash equilibrium will be a pure Nash equilibrium.

While the resulting equilibria will no longer be justified only by the requirement that all bidders play best responses, they will constitute a so-called *market equilibrium*: A situation in which every participant in the market receives the outcome that maximizes his utility and the market clears because all items are sold.

Indeed, while our analysis will follow Varian (2007) and has been done in a similar manner in Edelman et al. (2007), most of the results that we will discuss are actually well known results about market equilibria in the assignment game.

Note that in the GSP mechanism a bidder's payment is independent of his own bid and determined entirely by the slot that it receives. In a pure Nash equilibrium a bidder that is assigned slot s prefers this slot over any slot $t < s$ or $t > s$. We obtain the following characterization in terms of the per click payments.

Observation 11.7. *A bid profile b is a pure Nash equilibrium (PNE) if and only if for all bidders s,*

$$\alpha_s \cdot (v_s - p_s) \geq \alpha_t \cdot (v_s - p_t) \qquad \text{for all } t \geq s, \text{ and}$$
$$\alpha_s \cdot (v_s - p_s) \geq \alpha_t \cdot (v_s - p_{t-1}) \qquad \text{for all } t < s,$$

where we omitted the bids b from the formulas for notational convenience.

Note the asymmetry in this definition that stems from the fact that if bidder i targets a better slot $t < s$ then he has to pay a higher price than the bidder currently in that position, while he has to pay the same price as the bidder occupying the slot that he is targeting if he targets a worse slot $t > s$.

The following definition gets rid of this asymmetry. It requires that no bidder wants to swap bids with a different bidder.

Definition 11.8 (Varian, 2007)**.** *A bid profile b is a symmetric pure Nash (SNE) equilibrium if for all bidders s,*

$$\alpha_s \cdot (v_s - p_s) \geq \alpha_t \cdot (v_s - p_t) \qquad \text{for all } t,$$

where we again omitted the bids b from the formula for notational convenience.

Note that the prices in a SNE guarantee that every bidder, at the current prices, receives the slot that maximizes his utility.

3 A Simpler Necessary and Sufficient Condition

It turns out that the equilibrium conditions can be encapsulated in a simpler requirement. Instead of requiring that the inequalities hold for deviations to all possible slots it is sufficient to prevent deviations to the slot right above and right below.

Proposition 11.9. *If bids b satisfy the following two inequalities for all bidders s,*

$$\alpha_s \cdot (v_s - p_s) \geq \alpha_{s-1} \cdot (v_s - p_{s-1}), \qquad \text{and}$$
$$\alpha_s \cdot (v_s - p_s) \geq \alpha_{s+1} \cdot (v_s - p_{s+1})$$

then these bids are a symmetric Nash equilibrium.

Instead of proving this result formally we will give the basic idea by considering an example with three bidders and three slots. The only "long haul" deviations that are not covered are deviations from 1 to 3 and from 3 to 1. We will show that bidder 1 will not find it beneficial to deviate to slot 3. The argument for the opposite direction is similar.

We will first argue that $v_1 \geq v_2$. Since bidder 1 does not want to deviate to slot 2 and bidder 2 does not want to deviate to slot 1:

$$\alpha_1 \cdot (v_1 - p_1) \geq \alpha_2 \cdot (v_1 - p_2) \quad \text{and} \quad \alpha_2 \cdot (v_2 - p_2) \geq \alpha_1 \cdot (v_2 - p_1)$$

Adding these two inequalities we obtain $(\alpha_1 - \alpha_2)(v_1 - v_2) \geq 0$ which shows the claim.

Next we will use the fact that bidder 1 does not want to deviate to slot 2 and bidder 2 does not want to deviate to slot 3 together with the fact that $v_1 \geq v_2$ to conclude that bidder 1 does not want to deviate to slot 3. Namely,

$$\begin{aligned} \alpha_1 v_1 - \alpha_1 p_1 \geq \alpha_2 v_1 - \alpha_2 p_2 \quad &\Rightarrow \quad (\alpha_1 - \alpha_2) v_1 \geq \alpha_1 p_1 - \alpha_2 p_2 \\ \alpha_2 v_2 - \alpha_2 p_2 \geq \alpha_3 v_2 - \alpha_3 p_3 \quad &\Rightarrow \quad (\alpha_2 - \alpha_3) v_1 \geq \alpha_2 p_2 - \alpha_3 p_3, \end{aligned}$$

If we add up these two inequalities we obtain

$$(\alpha_1 - \alpha_3) v_1 \geq \alpha_1 p_1 - \alpha_3 p_3 \quad \Rightarrow \quad \alpha_1 v_1 - \alpha_1 p_1 \geq \alpha_3 v_1 - \alpha_3 p_3.$$

4 Characterization of Equilibrium Bids and Prices

As a next step we will obtain a characterization of the equilibrium bids and prices. It is this characterization that will allow us to explicitly solve for the symmetric pure Nash equilibrium with the lowest and the highest prices.

Since the agent in position s does not want to move down one slot and the agent in position $s+1$ does not want to move up one slot:

$$\begin{aligned} &\alpha_s(v_s - p_s) \geq \alpha_{s+1}(v_s - p_{s+1}), \qquad \text{and} \\ &\alpha_{s+1}(v_{s+1} - p_{s+1}) \geq \alpha_s(v_{s+1} - p_s). \end{aligned}$$

Combining these inequalities we obtain

$$(\alpha_s - \alpha_{s+1}) v_s + \alpha_{s+1} p_{s+1} \geq \alpha_s p_s \geq (\alpha_s - \alpha_{s+1}) v_{s+1} + \alpha_{s+1} p_{s+1}. \tag{1}$$

Recalling that $p_s = b_{s+1}$ we obtain

$$(\alpha_{s-1} - \alpha_s) v_{s-1} + \alpha_s b_{s+1} \geq \alpha_{s-1} b_s \geq (\alpha_{s-1} - \alpha_s) v_s + \alpha_s b_{s+1}. \tag{2}$$

Defining $x_s = \alpha_s / \alpha_{s-1} < 1$ we obtain

$$(1 - x_s) v_{s-1} + x_s b_{s+1} \geq b_s \geq (1 - x_s) v_s + \alpha_s b_{s+1}. \tag{3}$$

Inequalities (1) to (3) give three equivalent characterizations of the equilibrium bids showing that each agent's bid is bounded above and below by a convex combination of the bid by the agent right below him and either his own value or the value of the agent right above him. Symmetric pure Nash equilibria can be found by recursively choosing a sequence of bids that satisfy these inequalities.

5 Guarantees for Symmetric Equilibria

A first straightforward observation is that all symmetric pure Nash equilibria—unlike pure Nash equilibria— maximize social welfare.

Proposition 11.10. *Every symmetric pure Nash equilibrium maximizes social welfare.*

Proof. It suffices to show that $v_{s-1} \geq v_s$ for all s. By the definition of a symmetric Nash equilibrium we have

$$(\alpha_t - \alpha_s) \cdot v_t \geq \alpha_t \cdot p_t - \alpha_s \cdot p_s, \qquad \text{and}$$
$$(\alpha_s - \alpha_t) \cdot v_s \geq \alpha_s \cdot p_s - \alpha_t \cdot p_t.$$

By adding these inequalities we obtain

$$(\alpha_t - \alpha_s)(v_t - v_s) \geq 0,$$

which shows that $(v_t)_t$ and $(\alpha_t)_t$ must be ordered the same way. □

Next we use the characterization of the equiibrium bids to obtain the symmetric pure Nash equlibria with the lowest and highest revenue.

Theorem 11.11. *The lowest revenue symmetric pure Nash equilibrium yields the same revenue as the truthful VCG equilibrium.*

Proof. We obtain the revenue for the lowest revenue equilibrium by considering the lower boundary case of the inequalities in (2):

$$\alpha_{s-1} b_s^L = (\alpha_{s-1} - \alpha_s) v_s + \alpha_s b_{s+1}^L.$$

Solving this recursion yields

$$\alpha_{s-1} b_s^L = \sum_{t \geq s} (\alpha_{t-1} - \alpha_t) v_t.$$

The proof is concluded by observing that the left hand-side of this equality corresponds to the GSP price for slot $s-1$, while the right-hand side corresponds to the truthful VCG price for slot $s-1$. □

Theorem 11.12. *The highest revenue symmetric Nash equilibrium yields the same revenue as the highest revenue Nash equilibrium.*

Proof. Every symmetric pure Nash equilibrium is also a pure Nash equilibrium. This shows that the highest revenue in any pure Nash equilibrium can only be higher than the highest revenue in any symmetric Nash equilibrium. So to show equality, it suffices to show that the highest revenue in a symmetric pure Nash equilibrium is at least as high as the highest revenue in a pure Nash equilibrium.

To obtain the highest possible revenue in a symmetric pure Nash equilibrium we again consider the recursive characterization of equilibrium bids, each time choosing the highest possible bid. If we start from (2) we thus get

$$\alpha_{s-1} b_s^U = (\alpha_{s-1} - \alpha_s) v_{s-1} + \alpha_s b_{s+1}^U.$$

Using $p_s = b_{s+1}$,

$$\alpha_s p_s^U = (\alpha_s - \alpha_{s+1}) v_s + \alpha_{s+1} p_{s+1}^U.$$

On the other hand we can get a similar recursive formulation for the set of pure Nash equilibria. Namely, also in a pure Nash equilibrium the bidder in slot s does not want to deviate and target the next lower slot $s+1$. So,

$$\alpha_s p_s^N \leq (\alpha_s - \alpha_{s+1}) v_s + \alpha_{s+1} p_{s+1}^N.$$

Both recursions start at the bottommost slot $s = k$. Since the slot right below does not exist we can set $\alpha_{k+1} = 0$ and obtain

$$p_k^N \leq v_k = p_s^U.$$

Inspecting the above recursions we see that $p_s^U \geq p_s^N$ for all s. □

6 Concluding Remarks

There is a second line of work which identifies a different reason why the GSP mechanism may be the preferred mechanism in practice, which requires no strengthening of the equilibrium concept. This line of work starts from the observation that in practice the α_j's are not merely the click-through rate but rather represent a quality score, which is learned by the search engines. So a more realistic model is to assume that the scores α_j used by the mechanism are reasonably close to the true quality scores β_j, but that in general $\alpha \neq \beta$. It turns out that the VCG mechanism can fail to support the truthful VCG outcome even if the assumed qualities are only slightly off, while the GSP mechanisms is much more robust in this regard. More formally, one can show the following:

Theorem 11.13 (Dütting, Fischer, Parkes 2015)**.** *The set of α, β for which GSP preserves the truthful VCG outcome in some Nash equilibrium is a strict superset of the set of α, β's for which VCG achieves this.*

Recommended Literature

- Hal R. Varian. Position Auctions. International Journal of Industrial Organization, Vol. 25: 1163–1178, 2007. (Model, definition of a symmetric Nash equilibirum, most of the results)
- Benjamin Edelman, Michael Ostrovsky, Michael Schwarz. Internet Advertising and the Generalized Second Price Auction: Selling Billions of Dollars Worth of Keywords. American Economic Review, Vol.97(1):242–259, 2007. (Model, similar concept of a locally envy-free equilibrium)
- Cragiannis et al. Bounding the Inefficiency of Outcomes in Generalized Second Price Auctions. Journal of Economic Theory, Vol. 156: 343–388, 2015. (Price of Anarchy bounds for GSP)
- Paul Dütting, Felix Fischer, David C. Parkes. Truthful Outcomes from Non-Truthful Position Auctions. Working paper, 2015. (Increased robustness of non-truthful mechanisms)