Eidgenössische Technische Hochschule Zürich

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Datenstrukturen & Algorithmen — Exercise Sheet 2 — AS 16

Hand-in: Thursday, 6th October 2016 before the start of the lecture at 10:00 in the entrance area of ML D28. Please staple all sheets together and use this sheet as the cover page. Fill out the first two fields of the form below.

Exercise class (Room & Day): ________________

Submitted by: ________________

Corrected by: ________________

Bonus points: ____

Exercise 2.1 *Mathematical induction.*

Prove the following statements using mathematical induction over n.

a) $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

b) $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

c) $\sum_{k=0}^{n} \binom{r+k}{r} = \binom{r+n+1}{n}$ for each integer number $r \geq 0$

d) $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

Hint: The identities that you have seen in the lecture $\binom{n}{k} = \binom{n}{n-k}$ and $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ can be helpful.

Exercise 2.2 *Estimation using Integration.*

Prove the estimates from the lecture

$$\ln(n!) = n\ln(n) - n + \mathcal{O}(\ln n),$$

by writing $\ln(n!)$ as a sum and by estimating it upwards and downwards using integrals.

Hint: Use partial integration for integrating $\ln(x)$, respectively $\ln(x+1)$. It holds that $\ln(a \cdot b) = \ln(a) + \ln(b)$. Moreover you can use $\ln(x+1) \leq \ln(x) + \frac{1}{x}$ for $x > 0$ without a proof.

Please turn over.

Exercise 2.3 *Recurrence Relations.*

a) Let $f : \mathbb{N}_0 \to \mathbb{R}$ be an arbitrary function. Consider the recurrence relation

$$S(k) = \begin{cases} f(0) & \text{if } k = 0 \\ a \cdot S(k-1) + f(k) & \text{if } k \geq 1, \end{cases}$$

where we assume that a is constant and that $k \in \mathbb{N}_0$. Prove that $S(k) = \sum_{i=0}^{k} a^i f(k-i)$ using mathematical induction over k .

b) Use exercise part a), to solve the recurrence relation

$$T(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2T\left(\frac{n}{2}\right) + n & \text{if } n = 2^k \text{ for } k \in \mathbb{N} \end{cases}$$

Exercise 2.4 *Algorithm Design.*

You are in a skyscraper with 100 floors and want to find out, starting from which floor an egg breaks when thrown out of the window from there. If you throw an egg out of the window and it does not break, you can use it again. If it breaks, however, you cannot use it anymore. Naturally, if an egg breaks in floor i, it will also break when thrown from a floor $j > i$. On the other hand, if the egg does not break, it will not break when thrown from a floor $k < i$.

You have to develop a strategy that can find the wanted floor in every case and that minimizes the number of attempts (not the number of used eggs). You do not have to break all eggs. However, if your strategy breaks all eggs, you have to name the wanted floor number unambiguously and immediately after breaking the last egg.

a) Develop such a strategy under the assumption that you have an infinite number of eggs.

b) Develop such a strategy under the assumption that you have only *two* eggs.

   Hint: First consider what would be a good strategy if you only have one egg.