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Data Structures & Algorithm — Solutions to Sheet 2 — AS 16

Solution 2.1 *Mathematical Induction.*

a) *Base case* ($n = 1$): It holds that $\sum_{k=1}^{1} k^3 = 1^3 = 1 = \frac{1^2 \cdot 2^2}{4} = \frac{1^2(1+1)^2}{4}$.

Induction hypothesis: For some $n \in \mathbb{N}$: $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

Inductive step ($n \to n+1$):

$$\begin{aligned}\sum_{k=1}^{n+1} k^3 &= \Big(\sum_{k=1}^{n} k^3\Big) + (n+1)^3 \\ &\overset{I.H.}{=} \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{n^2(n+1)^2}{4} + \frac{4(n+1)(n+1)^2}{4} \\ &= \frac{(n^2+4n+4)(n+1)^2}{4} = \frac{(n+2)^2(n+1)^2}{4} = \frac{(n+1)^2((n+1)+1)^2}{4}. \quad \blacksquare\end{aligned}$$

b) *Base case* ($n = 1$): It holds that $(x+y)^1 = x + y = x^0y^1 + x^1y^0 = \binom{1}{0}x^0y^1 + \binom{1}{1}x^1y^0 = \sum_{k=0}^{1}\binom{1}{k}x^ky^{1-k}$.

Induction hypothesis: For some $n \in \mathbb{N}$: $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$.

Inductive step ($n \to n+1$):

$$\begin{aligned}(x+y)^{n+1} &= (x+y)(x+y)^n \\ &\overset{I.H.}{=} (x+y)\Big(\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}\Big) \\ &= \Big(\sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k}\Big) + \Big(\sum_{k=0}^{n}\binom{n}{k}x^ky^{n+1-k}\Big) \\ &= \Big(\sum_{k=1}^{n+1}\binom{n}{k-1}x^ky^{n+1-k}\Big) + \Big(\sum_{k=0}^{n}\binom{n}{k}x^ky^{n+1-k}\Big) \\ &= \binom{n}{0}x^0y^{(n+1)-0} + \sum_{k=1}^{n}\Big(\binom{n}{k-1}x^ky^{n+1-k} + \binom{n}{k}x^ky^{n+1-k}\Big) \\ &\quad + \binom{n}{n}x^{n+1}y^{(n+1)-(n+1)} \\ &= \binom{n+1}{0}x^0y^{(n+1)-0} + \sum_{k=1}^{n}\binom{n+1}{k}x^ky^{(n+1)-k} + \binom{n+1}{n+1}x^{n+1}y^{(n+1)-(n+1)} \\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}x^ky^{(n+1)-k}. \quad \blacksquare\end{aligned}$$

c) *Base case* ($n=1$): It holds that $\sum_{k=0}^{1}\binom{r+k}{r} = \binom{r}{r} + \binom{r+1}{r} = 1 + \frac{(r+1)!}{(r!)\cdot(1!)} = r+2 = \frac{(r+2)!}{((r+2-1)!)\cdot(1!)} = \binom{r+1+1}{1}$.

Induction hypothesis: For some $n \in \mathbb{N}$: $\sum_{k=0}^{n}\binom{r+k}{r} = \binom{r+n+1}{n}$.

Inductive step ($n \to n+1$):

$$\begin{aligned}\sum_{k=0}^{n+1}\binom{r+k}{r} &= \Big(\sum_{k=0}^{n}\binom{r+k}{r}\Big) + \binom{r+(n+1)}{r}\\ &\overset{I.H.}{=} \binom{r+n+1}{n} + \binom{r+n+1}{(r+n+1)-r}\\ &= \binom{r+n+1}{n} + \binom{r+n+1}{n+1} = \binom{r+(n+1)+1}{n+1}. \quad \blacksquare\end{aligned}$$

d) *Base case* ($n=1$): It holds that $(1+x)^1 = \binom{1}{0}x^0 + \binom{1}{1}x^1 = \sum_{k=0}^{n}\binom{n}{k}x^k$.

Induction hypothesis: For some $n \in \mathbb{N}$: $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$.

Inductive step ($n \to n+1$):

$$\begin{aligned}(1+x)^{n+1} &= (1+x)(1+x)^n\\ &\overset{I.H.}{=} (1+x)\sum_{k=0}^{n}\binom{n}{k}x^k\\ &= \Big(\sum_{k=0}^{n}\binom{n}{k}x^k\Big) + \Big(\sum_{k=0}^{n}\binom{n}{k}x^{k+1}\Big)\\ &= \Big(\sum_{k=0}^{n}\binom{n}{k}x^k\Big) + \Big(\sum_{k=1}^{n+1}\binom{n}{k-1}x^k\Big)\\ &= \binom{n}{0}x^0 + \sum_{k=1}^{n}\Big(\binom{n}{k}x^k + \binom{n}{k-1}x^k\Big) + \binom{n}{n}x^{n+1}\\ &= \binom{n+1}{0}x^0 + \sum_{k=1}^{n}\binom{n+1}{k}x^k + \binom{n+1}{n+1}x^{n+1} = \sum_{k=0}^{n+1}\binom{n+1}{k}x^k. \quad \blacksquare\end{aligned}$$

Solution 2.2 *Estimation using Integration.*

It holds that $\ln(n!) = \sum_{k=1}^{n}\ln(k) = \sum_{k=2}^{n}\ln(k)$ (the last equality holds, as $\ln(1)=0$), so we estimate the sum as shown in the lecture by

$$\int_1^n \ln(x)\ \mathrm{dx} \le \sum_{k=2}^{n}\ln(k) \le \int_1^n \ln(x+1)\ \mathrm{dx}$$

upwards and downwards. We compute the integrals with partial integration and obtain

$$\int_1^n \ln(x)\ \mathrm{dx} = \int_1^n 1\cdot\ln(x)\ \mathrm{dx} = \Big[x\ln x\Big]_1^n - \int_1^n x\cdot\frac{1}{x}\ \mathrm{dx} = n\ln n - 0 - n + 1$$

and

$$\begin{aligned}\int_1^n \ln(x+1)\ \mathrm{dx} &= \int_1^n 1\cdot\ln(x+1)\ \mathrm{dx} = \Big[(x+1)\ln(x+1)\Big]_1^n - \int_1^n (x+1)\cdot\frac{1}{x+1}\ \mathrm{dx}\\ &= (n+1)\ln(n+1) - 2\ln(2) - n + 1.\end{aligned}$$

Hence, we get

$$
\begin{aligned}
n \ln n - n + 1 \leq \ln(n!) &\leq (n+1)\ln(n+1) - 2\ln(2) - n + 1 \\
&= n\ln(n+1) + \ln(n+1) - 2\ln(2) - n + 1 \\
&\leq n\ln(n) + 1 + \ln(n+1) - 2\ln(2) - n + 1 = n\ln(n) - n + \mathcal{O}(\ln n),
\end{aligned}
$$

and it follows that $\ln(n!) = n \ln n - n + \mathcal{O}(\ln n)$. ∎

Solution 2.3 *Recurrence Relations.*

a) *Base case* $(k = 0)$: It holds that $S(0) = f(0) = a^0 f(0-0) = \sum_{i=0}^{0} a^i f(0-i)$.

Induction hypothesis: For some $k \in \mathbb{N}$: $S(k) = \sum_{i=0}^{k} a^i f(k-i)$.

Inductive step $(k \to k+1)$:

$$
\begin{aligned}
S(k+1) &= a \cdot S(k) + f(k+1) \\
&\overset{I.H.}{=} a\Big(\sum_{i=0}^{k} a^i f(k-i)\Big) + f(k+1) \\
&= \Big(\sum_{i=0}^{k} a^{i+1} f(k-i)\Big) + a^0 f(k+1) \\
&= \Big(\sum_{i=1}^{k+1} a^i f(k-i+1)\Big) + a^0 f((k+1)-0) = \sum_{i=0}^{k+1} a^i f((k+1)-i). \quad ∎
\end{aligned}
$$

b) Let $n = 2^k$ for some $k \in \mathbb{N}_0$. We set $f(k) = 2^k$, $a = 2$ and

$$
T'(k) = \begin{cases} 1 & \text{if } k = 0 \\ 2T'(k-1) + 2^k & \text{if } k \geq 1, \end{cases}
$$

which corresponds exactly to the form in exercise part a). The solution is

$$
\begin{aligned}
T'(k) &= \sum_{i=0}^{k} 2^i f(k-i) \\
&= \sum_{i=0}^{k} 2^i \cdot 2^{k-i} = 2^k + k \cdot 2^k = (k+1) \cdot 2^k.
\end{aligned}
$$

We note that $T'(k) = T(n)$. Therefore, we obtain $T(n) = (\log_2(n) + 1) \cdot n \in \Theta(n \log n)$ using the above formula.

Solution 2.4 *Algorithm Design.*

a) We throw the first egg out the window in the 50th floor. If it breaks, we know that the floor of interest is one of the first fifty floor. Otherwise, we know that the floor of interest is between the 51st and the 100th floor. In both cases, we only have to consider 50 floors. Then, we continue in the same way by throwing an egg (if necessary a new one) from the 25th, respectively the 75th floor. More formally, our strategy would be the following:

1) Set $U \leftarrow 1$ and $O \leftarrow 100$. These variables indicate, which floor we will search at least (U) and at most (O).

2) While $O > U$:

3) Compute $S \leftarrow \lfloor (U + O)/2 \rfloor$.

4) Throw the egg out of the window in floor S.

5) If the egg breaks, the floor we are looking for is between U and S. Hence, we set $O \leftarrow S$.

6) If the egg does not break, the floor we are looking for is between $S + 1$ and O. Hence, we set $U \leftarrow S + 1$.

7) As the lower bound U and the upper bound O are equal, the floor of interest is $U = O$.

We can immediately see that the strategy will need $\Theta(\log(n))$ attempts to find the floor we are looking for in a building with n floors. For $n = 100$ one can show that the strategy will never need more than seven attempts (and always at most six).

b) If we would only have one egg, we would need to test every single floor from bottom to top. Having a second egg, we can restrict the search space in advance. We proceed as follows: We throw one egg out of the window in some floor s, which we will determine later. If it breaks, we know that the floor of interest is somewhere between 1 and s and we use the second egg to find the floor as discussed above. On the other hand, if the egg does not break, we can continue at floor $s + (s - 1)$. If the egg breaks in this second attempt, the floor of interest is between floor $s + 1$ and floor $s + (s - 1)$ and we use the second egg to find the floor in this range. Otherwise, we continue with floor $s + (s - 1) + (s - 2)$ and so forth. It remains to determine s. As the building has 100 floors, it must hold that

$$100 \leq s + (s - 1) + (s - 2) + \cdots + 2 + 1 = \sum_{k=1}^{s} k = \frac{s(s + 1)}{2}.$$

The smallest natural number s that satisfies the inequality is $s = 14$. Therefore, we throw the first egg down from floor 14th floor, then from the 27th floor, then from the 39th floor and so forth until the first egg breaks. We observe: If the first egg breaks after i attempts, $s - i$ additional attempts are sufficient to find the floor of interest. Therefore, s attempts in total are always enough to find the floor we are looking for, i.e. 14 attempts for a building with 100 floors.