1 Union bound

A very important tool in discrete probability theory is the so called union bound. This bound will occur numerous times throughout the course. We state it without giving a proof here.

**Theorem 1.** Let \((\Omega, \Pr)\) be a discrete probability space and let \(A_1, A_2, \ldots, A_n \subseteq \Omega\) be events. Then we have

\[
\Pr \left[ \bigcup_{i=1}^{n} A_i \right] \leq \sum_{i=1}^{n} \Pr[A_i].
\]

2 Landau Symbols / \(\mathcal{O}\)-Notation

Throughout the course we will use the so called Landau symbols to describe asymptotic behavior of functions. For two functions, \(f, g : \mathbb{N} \to \mathbb{R}\) we write

\[
f = \mathcal{O}(g) \quad \text{if } 0 \leq \limsup_{n \to \infty} \left| \frac{f(n)}{g(n)} \right| < \infty,
\]

\[
f = o(g) \text{ or } f \ll g \quad \text{if } \lim_{n \to \infty} \left| \frac{f(n)}{g(n)} \right| = 0,
\]

\[
f = \Omega(g) \quad \text{if } 0 < \liminf_{n \to \infty} \left| \frac{f(n)}{g(n)} \right| \leq \infty,
\]

\[
f = \omega(g) \text{ or } f \gg g \quad \text{if } \lim_{n \to \infty} \left| \frac{f(n)}{g(n)} \right| = \infty,
\]

\[
f = \Theta(g) \quad \text{if } f = \Omega(g) \text{ and } f = \mathcal{O}(n).
\]

An alternative equivalent definition for \(\mathcal{O}, \Omega\) and \(\Theta\) is

\[
f = \mathcal{O}(g) \quad \text{if } \exists C > 0, n_0 : \forall n \geq n_0 : |f(n)| \leq C|g(n)|,
\]

\[
f = \Omega(g) \quad \text{if } \exists c > 0, n_0 : \forall n \geq n_0 : |f(n)| \geq c|g(n)|,
\]

\[
f = \Theta(g) \quad \text{if } \exists c > 0, C > 0, n_0 : \forall n \geq n_0 : c|g(n)| \leq |f(n)| \leq C|g(n)|.
\]

2.1 Examples

- \(1000n = \Theta(n)\),
- \(n = o(n^{1+\varepsilon})\) for every \(\varepsilon > 0\),
- \(n^{100} = o(\log(n)^{\log(n)})\) since \(\log(n)^{\log(n)} = n^{\log \log(n)}\),
- \(\log(n)^{\delta} = o(n^{\varepsilon})\) for all constants \(\delta, \varepsilon > 0\) since \(\log(n)^{\delta} = e^{\delta \log \log n}\) and \(n^{\varepsilon} = e^{\varepsilon \log n}\),
\[
e^{-\Omega(n)} = o(1), \text{ but not } (!!) \quad e^{\Omega(n)} = O(1) \text{ since the first equality is not correct since the denominator might be of order of magnitude smaller than the numerator in which case the exponent tends to infinity! The second equality is correct though.}
\]

3 Law of Total Probability / Law of Total Expectation

A very useful tool in discrete probability are the laws of total probability and total expectation.

**Theorem 2.** Let \((\Omega, \Pr)\) be a discrete probability space and let \(A_1, A_2, \ldots, A_n \subseteq \Omega\) be events that form a partition of \(\Omega\), that is, the events are pairwise disjoint and their union is \(\Omega\). Then we have for every event \(E \subseteq \Omega\) that

\[
\Pr[E] = \sum_{i=1}^{n} \Pr[E|A_i] \Pr[A_i],
\]

and for every random variable \(X\) that

\[
E[X] = \sum_{i=1}^{n} E[X|A_i] \Pr[A_i].
\]

**Proof.** We first prove that law of total probability. Note that since the \(A_i\)’s are pairwise disjoint, the events \(E \cap A_1, \ldots, E \cap A_n\) are also pairwise disjoint. Hence, we have

\[
\sum_{i=1}^{n} \Pr[E \cap A_i] = \Pr\left[\bigcup_{i=1}^{n} E \cap A_i\right]. \quad (1)
\]

With this we can derive

\[
\Pr[E] = \sum_{i=1}^{n} \Pr[E|A_i] \Pr[A_i] = \sum_{i=1}^{n} \frac{\Pr[E \cap A_i]}{\Pr[A_i]} \Pr[A_i] \overset{(1)}{=} \Pr\left[\bigcup_{i=1}^{n} E \cap A_i\right] = \Pr[E],
\]

where the last step follows from the fact that the union of the \(A_i\)’s equals \(\Omega\).
It remains to prove the law of total expectation. We have

\[
\sum_{i=1}^{n} \Pr[A_i] \mathbb{E}[X | A_i] = \sum_{i=1}^{n} \Pr[A_i] \sum_{x \in \Omega} x \Pr[X = x | A_i]
\]

\[
= \sum_{i=1}^{n} \sum_{x \in \Omega} x \Pr[X = x | A_i] \Pr[A_i]
\]

\[
= \sum_{x \in \Omega} \sum_{i=1}^{n} x \Pr[X = x | A_i] \Pr[A_i]
\]

\[
= \sum_{x \in \Omega} x \sum_{i=1}^{n} \Pr[X = x | A_i] \Pr[A_i]
\]

\[
= \sum_{x \in \Omega} x \sum_{i=1}^{n} \Pr[X = x \land A_i]
\]

\[
= \sum_{x \in \Omega} x \Pr[X = x] = \mathbb{E}[X].
\]

\[\square\]
4 Useful Inequalities

Inequality 3. For all $x \in \mathbb{R}$ we have

$$1 - x \leq e^{-x}.$$  

Proof. We just give an informal picture proof.

![Graph showing $e^{-x}$ and $1 - x$]

Inequality 4. For all $n \in \mathbb{N}$, $k \in \mathbb{N}_0$ with $k \leq n$ we have

(i) $\binom{n}{k} \leq 2^n$,

(ii) $\binom{n}{k} \leq \frac{n^k}{k!}$, and

(iii) $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$.

Proof. (i) We have

$$\binom{n}{k} \leq \sum_{i=0}^{n} \binom{n}{k} = 2^n.$$  

(ii) We have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \cdots (n-k+1)}{k!} \leq \frac{n^k}{k!}.$$  

(iii) We first $\binom{n}{k} \geq (n/k)^k$. We have

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1) \cdots (n-k+1)}{k(k-1) \cdots 1} = \prod_{i=0}^{k-1} \frac{n-i}{k-i} \geq \left(\frac{n}{k}\right)^k,$$

where the last inequality follows from the fact that for every $k \leq n$ we have $(n-i)/(k-i) \geq n/k$. It remains to show $\binom{n}{k} \leq (ne/k)^k$. We have

$$e^k = \sum_{i=0}^{\infty} \frac{k^i}{i!} \geq \frac{n(n-1) \cdots (n-(k-1))}{k!} \cdot \frac{k^k}{n(n-1) \cdots (n-(k-1))} \geq \binom{n}{k} \left(\frac{k}{n}\right)^k,$$
which immediately implies the claim.

\[ \ln n \leq H_n \leq \ln n + 1. \]

**Proof.** Recall that \( \ln = \int_1^n 1/x \, dx \). The following picture illustrates that \( H_n \leq \ln(n) + 1 \) and, by shifting the \( 1/x \)-curve one unit to the left, that \( \ln(n-1) \leq H_n \).

In fact, one can show that \( H_n = \ln n + \gamma + O(n^{-1}) \) where \( \gamma \approx 0.5772 \) denotes the Euler-Mascheroni constant. \( \square \)