Exam: Randomized Algorithms and Probabilistic Methods

Note: The following equality might be useful:
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \]

Exercise 1 \hspace{1cm} (6 points)
Of the \(2n\) people in a given collection of \(n\) couples, exactly \(m\) die. Assuming the \(m\) have been picked uniformly at random find the expected number of surviving couples.

Exercise 2 \hspace{1cm} (6 points)
A 3-way-cut of an undirected graph \(G = (V,E)\) is a partition of the vertices into three disjoint parts \(S \cup T \cup U = V\). The size of such a cut is defined as the number of edges whose endpoints lie in different parts of the partition.

1. Show that any graph with \(m\) edges has a 3-way cut of size at least \(2m/3\).
2. Give a polynomial time deterministic algorithm for finding such a cut.

Exercise 3 \hspace{1cm} (6 points)
Prove that with probability \(1 - o(1)\) every pair of vertices in \(G_{n,\frac{1}{2}}\) is connected by at least
\[ \frac{n - 2}{4} - \sqrt{(n - 2) \log(n\alpha(n))} \]
paths of length 2 where \(\alpha(n) = \omega(1)\).
Exercise 4  
(6 points)

By $[n]_p$, we denote a random subset of $[n] = \{1, 2, \ldots, n\}$ in which every element is contained with probability $p$ independently of all other elements. For a constant $k \in \mathbb{N}$ find a function $p_k(n)$ which is a weak threshold for the appearance of $k$ consecutive elements in $[n]_{p_k(n)}$.

Exercise 5  
(12 points)

Let $G = G(n, p)$ with $p = \left(\frac{C \log n}{n^2}\right)^{1/3}$ where $C > 0$ is a constant.

1. For a pair of vertices $u, v \in V$ compute the expected number of paths of length 3 connecting $u$ and $v$ exactly.

2. Prove that there exists a constant $C > 0$ such that with high probability every pair of vertices is connected with a path of length 3.

Exercise 6  
(12 points)

In the lecture and on the fourth graded homework sheet you have learned about two ways to rapidly shuffle cards. Now consider the following card shuffling method. In each step, select card at position $i$ uniformly at random and swap it with the card at position $j$ chosen uniformly at random ($i$ and $j$ can be the same card). You may assume that the chain is irreducible, aperiodic and that the stationary distribution is the uniform distribution.

Generate a coupling for the process and prove that the expected time until it synchronizes is in time $O(n^2)$ for all possible starting states.

Exercise 7  
(12 points)

Consider a biased random walk on the integers starting at 0 such that the probability of increasing is $p < \frac{1}{2}$ and the probability of decreasing is $(1 - p)$. For a given constant $k \in \mathbb{N}$ what is the probability of ever reaching $k$?

Hint. You may assume that the probability is not 1.

Good Luck!