Sample Exam:
Randomized Algorithms and Probabilistic Methods

Exercise 1
A fair coin is flipped over and over. Calculate the expected number of coin flips until three consecutive tails or three consecutive heads appear.

(10 points)

Exercise 2
Let $G$ be a graph with $m$ edges and $n$ vertices. We color the edges of $G$ with $k \geq 6$ colors uniformly at random independently from each other. Let $X$ denote the number of edges in the majority color (i.e., in the color that is assigned to most edges). Prove that

$$\Pr \left[ X \geq \frac{m}{k} + \sqrt{m \ln k} \right] \leq e^{-k/6} .$$

(10 points)

Exercise 3
Consider a set $N$ of size $|N| = n$ and a fixed subset $A \subseteq N$ of size $|A| = a$. We assume that $n$ is even.

We partition $N$ randomly into pairs. That is, all possible partitions of $N$ into $n/2$ sets of size 2 appear with the same probability. Let the random variable $X$ denote the number of elements in $A$ that are paired with another element of $A$. (Thus $X$ is always even by definition.)

Calculate $E[X]$ and $\text{Var}[X]$ exactly.

(10 points)

Hint: How would you pick such a random partition sequentially?

Exercise 4
Consider the complete graph $K_n$ on $n$ vertices. We color its edges with 3 colors uniformly at random independently from each other. Show that the probability that the colored graph does not contain a monochromatic triangle is bounded by $e^{-Cn^2}$ for some constant $C > 0$ if $n$ is large enough. (You don’t have to specify $C$ explicitly, just prove its existence.)

(10 points)
**Exercise 5**

We throw $m = m(n)$ balls into $n$ bins uniformly at random independently from each other. Find and prove a weak threshold for the property $\mathcal{P} := \text{’There is a bin containing two or more balls’}$. I.e., find a function $m_0(n)$ such that

$$\lim_{n \to \infty} \Pr[\mathcal{P}] = \begin{cases} 0, & \text{if } m(n) \ll m_0(n), \\ 1, & \text{if } m(n) \gg m_0(n). \end{cases}$$

(15 points)

**Exercise 6**

Let $G = (V, E)$ be a graph with maximum degree $\Delta \geq 1$. It is well-known (and easy to show) that every such graph has a proper vertex-coloring with $\Delta + 1$ colors, i.e., a coloring $c : V \to \{1, \ldots, \Delta + 1\}$ with $c(u) \neq c(v)$ if $\{u, v\} \in E$. In this exercise, we consider colorings with $k := 2\Delta + 2$ colors.

Consider the following Markov chain: The states are all proper vertex-colorings of $G$ with $k$ colors. In every transition, a vertex $v \in V$ is chosen uniformly at random and assigned a new color uniformly at random from the set of all colors that do not appear in the neighborhood of $v$ and are different from the current color of $v$ (i.e., from the set of all colors that produce a proper $k$-vertex-coloring of $G$ that is different from the current one).

Prove that this Markov chain converges to the uniform distribution, regardless of the starting distribution.

(15 points)

**Good Luck!**