Additive Spanners: A Simple Construction

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Abstract. We consider additive spanners of unweighted undirected graphs. Let G be a graph and H a subgraph of G. The most naïve way to construct an additive k-spanner of G is the following: As long as H is not an additive k-spanner repeat: Find a pair $(u, v) \in H$ that violates the spanner-condition and a shortest path from u to v in G. Add the edges of this path to H.

We show that, with a very simple initial graph H, this naïve method gives additive 6- and 2-spanners of sizes matching the best known upper bounds. For additive 2-spanners we start with $H = \emptyset$ and end with $O(n^{3/2})$ edges in the spanner. For additive 6-spanners we start with H containing $\lfloor n^{1/3} \rfloor$ arbitrary edges incident to each node and end with a spanner of size $O(n^{4/3})$.

1 Introduction

Additive spanners are subgraphs that preserve the distances in the graph up to an additive positive constant. Given an unweighted undirected graph G, a subgraph H is an additive k-spanner if for every pair of nodes u, v it is true that

$$d_G(u, v) \leqslant d_H(u, v) \leqslant d_G(u, v) + k$$

In this paper we only consider purely additive spanners, which are k-spanners where $k = O(1)$. Throughout this paper every graph will be unweighted and undirected.

Many people have considered a variant of this problem, namely multiplicative spanners and even mixes between additive and multiplicative spanners [1,2,3]. The problem of finding a k-spanner of smallest size has received a lot of attention. Most notably, given a graph with n nodes Dor et al. [4] prove that it has a 2-spanner of size $O(n^{3/2})$, Baswana et al. [5] prove that it has a 6-spanner of size $O(n^{4/3})$, and Chechik [6] proves that it has a 4-spanner of size $O(n^{7/5} \log^{1/5} n)$. Woodrufff [7] shows that for every constant k there exist graphs with n nodes such that every $(2k-1)$-spanner must have at least $\Omega(n^{1+1/k})$ edges. This implies that the construction of 2-spanners are optimal. Whether there exists an algorithm for constructing $O(1)$-spanners with $O(n^{1+\varepsilon})$ edges for some $\varepsilon < 1/3$ is unknown and is an important open problem.

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Let G be a graph and H a subgraph of G. Consider the following algorithm: As long as there exists a pair of nodes u, v such that $d_H(u,v) > d_G(u,v) + k$, find a shortest path from u to v in G and add the edges on the path to H. This process will be referred to as **k-spanner-completion**. After k-spanner-completion, H will be a k-spanner of G. Thus, given a graph G, a general way to construct a k-spanner for G is the following: Firstly, find a simple subgraph of G. Secondly use k-spanner-completion on this subgraph. The main contribution of this paper is:

Theorem 1. *Let G be a graph with n nodes and H the subgraph containing all nodes but no edges of G. For each node add $\lfloor n^{1/3} \rfloor$ edges adjacent to that node to H (or, if the degree is less, add all edges incident to that node). After 6-spanner-completion H will have at most $O(n^{4/3})$ edges.*

It is well-known that a graph with n nodes has a 6-spanner of size $O(n^{4/3})$ [5]. The techniques employed in our proof of correctness are similar to those in [5]. The creation of the initial graph H corresponds to the clustering in [5] and the 6-spanner-completion corresponds to their path-buying algorithm. For completeness we show that the same method gives a 2-spanner of size $O(n^{3/2})$. This fact is already known due to [4] and is matched by a lower bound from [7].

Theorem 2. *Let G be a graph with n nodes and H the subgraph where all edges are removed. Upon 2-spanner-completion H has at most $O(n^{3/2})$ edges.*

2 Creating a 6-Spanner

The algorithm for creating a 6-spanner was described in the abstract and the introduction.

For a given graph G, a 6-spanner of G can be created by strating with some subgraph H of G and applying 6-spanner-completion to H. Theorem 1 states that for a suitable starting choice of H we get a spanner of size $O(n^{4/3})$. The purpose of this section is to show that the 6-spanner created has no more than $O(n^{4/3})$ edges. This will imply that the construction (in terms of the size of the 6-spanner) matches the best known upper bound [5].

Proof (of Theorem 1). Inserting (at most) $\lfloor n^{1/3} \rfloor$ edges per node will only add $n \lfloor n^{1/3} \rfloor = O(n^{4/3})$ edges to H. Therefore it is only necessary to prove that 6-spanner-completion adds no more than $O(n^{4/3})$ edges.

Let $v(H)$ and $c(H)$ be defined by:

$$v(H) = \sum_{u,v \in V(G)} \max\{0, d_G(u,v) - d_H(u,v) + 5\}, \quad c(H) = \#E(H)$$

Say that a shortest path, p, from u to v is added to H, and let H_0 be the subgraph before the edges are added. Let the path consist of the nodes:

$$u = w_0, w_1, \ldots, w_r = v, r \in \mathbb{N}$$

Let $u' = w_i$ be the node w_i with the smallest i such that $\deg_{H_0}(w_i) \geqslant \lfloor n^{1/3} \rfloor$. Likewise let $v' = w_j$ be the node w_j the largest j such that $\deg_{H_0}(w_j) \geqslant \lfloor n^{1/3} \rfloor$. Remember that if $\deg_{H_0}(w_i) < \lfloor n^{1/3} \rfloor$ then all the edges adjacent to w_i are already in H_0. This implies that $d_{H_0}(u', v') > d_G(u', v') + 6$ since $d_{H_0}(u, v) > d_G(u, v) + 6$.

Say that t new edges are added to H. Then there must be at least t nodes on p with degree $> n^{1/3}$. Since every node can be adjacent to no more than 3 nodes on p (since it is a shortest path) there must be $\Omega(n^{1/3}t)$ nodes adjacent to p in H. Let z and w be neighbours to u' and v' in H respectively and let r be any node adjacent to p in H. Let s be a node on p such that r and s are adjacent in H. See Figure 1 for an illustration.

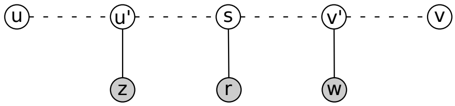

Fig. 1. The dashed line denotes the shortest path from u to v. The solid lines denote edges.

By the triangle inequality we see that:

$$d_H(z, r) + d_H(r, w) \leqslant d_G(u', v') + 4$$

But on the other hand:

$$d_{H_0}(z, r) + d_{H_0}(r, w) \geqslant d_{H_0}(z, w) \geqslant d_{H_0}(u', v') - 2 > d_G(u', v') + 4$$

Combining these two inequalities we obtain $d_{H_0}(z, r) > d_H(z, r)$ or $d_{H_0}(r, w) > d_H(r, w)$. And from the triangle inequality $d_G(z, r) + 5 > d_H(z, r)$ and $d_G(r, w) + 5 > d_H(r, w)$. Since u' and v' have at least $n^{1/3}$ neighbours and there are $\Omega(n^{1/3}t)$ nodes in H adjacent to p, the definition of $v(H)$ implies that:

$$v(H) - v(H_0) \geqslant \Omega(t(n^{1/3})^2)$$

And since $c(H) - c(H_0) = t$:

$$\frac{v(H) - v(H_0)}{c(H) - c(H_0)} \geqslant \Omega(n^{2/3})$$

Since $v(H) \leqslant O(n^2)$ this implies that $c(H)$ increases with no more than $O(n^2/n^{2/3}) = O(n^{4/3})$ in total when all shortest paths are inserted. Hence $c(H) = O(n^{4/3})$ when the 6-spanner-completion is finished which yields the conclusion. □

3 Creating a 2-Spanner

For completeness we show that 2-spanner-completion gives spanners with $O(n^{3/2})$ edges. This matches the upper bound from [4] and the lower bound from [7].

Proof (of Theorem 2). Let G be a graph with n nodes. Whenever H is a spanner of G, define $v(H)$ and $c(H)$ as:

$$v(H) = \sum_{u,v \in V(G)} \max\{0, d_G(u,v) - d_H(u,v) + 3\}, \quad c(H) = \sum_{v \in V(G)} (\deg_H(v))^2$$

It is easy to see that $0 \leqslant v(H) \leqslant 3n^2$ and by Cauchy-Schwartz's inequality $\sqrt{c(H) \cdot n} \geqslant 2\#E(H)$. The goal will be to prove that when the algorithm terminates $c(H) = O(n^2)$, since this implies that $\#E(H) = O(n^{3/2})$. This is done by proving that in each step of the algorithm $c(H) - 12v(H)$ will not increase. Since $v(H) = O(n^2)$ this means that $c(H) = O(n^2)$ which ends the proof. Therefore it is sufficient to check that $c(H) - 12v(H)$ never increases.

Consider a step where new edges are added to H on a shortest path from u to v of length t. Let H_0 be the subgraph before the edges are added. Assume that u, v violates the 2-spanner condition in H_0, i.e. $d_{H_0}(u,v) > d_G(u,v) + 2$. Let the shortest path consist of the nodes:

$$u = w_0, w_1, \ldots, w_{t-1}, w_t = v$$

It is obvious that:

$$c(H) - c(H_0) \leqslant \sum_{i=0}^{t} (\deg_H(w_i))^2 - (\deg_H(w_i) - 2)^2 \leqslant 4 \sum_{i=0}^{t} \deg_H(w_i)$$

Every node cannot be adjacent to more than 3 nodes on the shortest path, since otherwise it would not be a shortest path. Using this insight we can bound the number of nodes which in H are adjacent to or on the shortest path from below by:

$$\frac{1}{3} \sum_{i=0}^{t} \deg_H(w_i)$$

Now let z be a node in H adjacent or on to the shortest path. Obviously:

$$d_H(u,z) + d_H(z,v) \leqslant d_G(u,v) + 2$$

Furthermore $d_{H_0}(u,z) + d_{H_0}(z,v) > d_G(u,v) + 2$ since otherwise there would exist a path from u to v in H_0 of length $\leqslant d_G(u,v) + 2$. Hence:

$$d_H(u,z) + d_H(z,v) < d_{H_0}(u,z) + d_{H_0}(z,v)$$

Now let z be a node on the shortest path which is adjacent to w_i in H (every node on the path will also be adjacent in H to such a node). Then by the triangle inequality:

$$\begin{aligned} d_H(u,z) \leqslant d_H(u,w_i) + d_H(w_i,z) &= d_G(u,w_i) + 1 \\ \leqslant d_G(u,z) + d_G(z,w_i) + 1 &= d_G(u,z) + 2 \end{aligned}$$

And likewise $d_H(z,v) \leqslant d_G(z,v) + 2$. Combining these two observations yields:

$$\sum_{w \in V} \max\{0, d_G(z,w) - d_H(z,w) + 3\} < \sum_{w \in V} \max\{0, d_G(z,w) - d_{H_0}(z,w) + 3\}$$

Since this holds for every node in H adjacent to or on the shortest path this means that:

$$v(H) - v(H_0) \geqslant \frac{1}{3} \sum_{i=0}^{t} \deg_H(w_i)$$

Combining this with the bound on $c(H) - c(H_0)$ gives:

$$(c(H) - 12v(H)) - (c(H_0) - 12v(H_0)) \leqslant 0$$

which finishes the proof. □

References

1. Pettie, S.: Low distortion spanners. In: Arge, L., Cachin, C., Jurdziński, T., Tarlecki, A. (eds.) ICALP 2007. LNCS, vol. 4596, pp. 78–89. Springer, Heidelberg (2007)
2. Elkin, M., Peleg, D.: $(1+\varepsilon, \beta)$-spanner constructions for general graphs. SIAM Journal on Computing 33(3), 608–631 (2004); See also STOC 2001
3. Thorup, M., Zwick, U.: Spanners and emulators with sublinear distance errors. In: Proc. 17th ACM/SIAM Symposium on Discrete Algorithms (SODA), pp. 802–809 (2006)
4. Dor, D., Halperin, S., Zwick, U.: All-pairs almost shortest paths. SIAM Journal on Computing 29(5), 1740–1759 (2000); See also FOCS 1996
5. Baswana, S., Kavitha, T., Mehlhorn, K., Pettie, S.: New constructions of (α, β)-spanners and purely additive spanners. In: Proc. 16th ACM/SIAM Symposium on Discrete Algorithms (SODA), pp. 672–681 (2005)
6. Chechik, S.: New additive spanners. In: Proc. 24th ACM/SIAM Symposium on Discrete Algorithms (SODA), pp. 498–512 (2013)
7. Woodruff, D.P.: Lower bounds for additive spanners, emulators, and more. In: Proc. 47th IEEE Symposium on Foundations of Computer Science (FOCS), pp. 389–398 (2006)